Signals, systems, acoustics and the ear

Week 3

Frequency characterisations of systems & signals
The big idea
As long as we know what the system does to sinusoids...

... we can predict any output to any input.
Representing signals as sums of sinusoids:
Spectra
Synthesising waves

French mathematician
Jean Baptiste Joseph Fourier
1768-1830
Fourier Synthesis

we add up sinewaves by adding up the respective amplitude values of all sine waves for each point in time

sinewave I: 200 Hz

+ 

sinewave II: 400 Hz

this leads to a complex waveform consisting of a 200 and a 400 Hz sinusoid
Beats: Add 2 sinewaves that are close in frequency

500 Hz

501 Hz
500, 501 Hz

500+501 Hz
Fourier Analysis

Suppose we are given a complex waveform:

The question is, which are the underlying sine waves?

Fourier series analysis (calculus based)
Fourier Analysis

What if the complex wave is really complex?
Fourier Analysis

plotted as a spectrum

etc.
How to determine a spectrum

• Easy to see how to *synthesise*
  – spectrum $\rightarrow$ waveform

• But how do we analyse?
  – waveform $\rightarrow$ spectrum

• A special case: periodic complex waves
  – All component sine waves must be *harmonically* related
  – Their frequencies must be integer (whole-number) multiples of the repetition frequency of the complex waveform
Adding more than two sinusoids: component sine waves

400 Hz
½ V

200 Hz
1 V
Adding Waveforms

sinusoids

complex waveform
Adding a third sinusoid

resulting periodic complex wave
Adding 15 sinusoids

with 15 sinusoids:

The resulting waveform becomes more and more like a sawtooth.
Spectrum of the sawtooth waveform

![Graph showing the spectrum of a sawtooth waveform with a 10 ms time scale and an amplitude spectrum showing the first 10 harmonics. The graph includes a table with frequencies and corresponding amplitudes for harmonics ranging from 100 to n x 100, where n = 1 to infinity. Each harmonic's amplitude is given by 1/n.](image-url)
Visual effects of 'phase'

Phase can have a great effect on the resulting complex waveform, e.g.:

200, 400, and 600 Hz sinusoids added:

all in the same (sine) phase

400 Hz sinusoid is + 90°
Other periodic complex waves

• Infinite number of possible periodic complex wave shapes.
• All complex periodic waves have spectra whose sine-wave components are harmonically-related
  – frequencies are whole-number (integer) multiples of a common “fundamental” frequency.
Vowel with fixed $f_0$

spectrum shows harmonics of 150 Hz

waveform repeats each $1/150$ s
What does the spectrum of a sinusoid look like?

**Waveform**

- **SINE**
- **SQUARE**
- **TRIANGULAR**

The spectrum of a sinusoid includes harmonics at integer multiples of the fundamental frequency ($F$), with amplitudes given by

1. \( \frac{1}{3}(3F) + \frac{1}{5}(5F) + \frac{1}{7}(7F) + \ldots \)
2. \( \frac{1}{9}(3F) + \frac{1}{25}(5F) + \frac{1}{49}(7F) + \ldots \)

The amplitude spectrum is shown with peaks at these harmonic frequencies.
Spectrum of a pulse train

the original

the approximation
Spectra of periodic waves

• Only the possible frequencies are constrained. The amplitude and phase of each harmonic can have any possible value
  – including zero amplitude.

• Fundamental frequency (f0) is the greatest common factor of harmonic frequencies.

• Series of harmonics at:
  – 100, 200, 300 Hz: f0 = 100Hz
  – 150, 200, 250 Hz: f0 = 50Hz
  – 200, 700, 1000 Hz: f0 = 100Hz
Spectra of aperiodic waves

- Aperiodic waves can also be constructed from a series of sinusoids ...
  - but not using harmonics only.
- Spectra are continuous – every possible frequency is present...
  - as if harmonics were infinitely close together.
- What is the spectrum of a single pulse?
Keep lowering the fundamental frequency of a train of pulses

100 Hz pulse train

50 Hz pulse train

Single pulse
Spectra of random aperiodic sounds

Q: Why ‘white’ and ‘pink’?
Q: Why ‘white’ and ‘pink’?
A: analogies to light waves

<table>
<thead>
<tr>
<th>Unit</th>
<th>Symbol</th>
<th>Value</th>
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<tr>
<td>kilo-</td>
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</tr>
<tr>
<td>mega-</td>
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<td>$10^{12}$</td>
</tr>
<tr>
<td>peta-</td>
<td>P</td>
<td>$10^{15}$</td>
</tr>
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</table>
Key Points

• Fourier synthesis
  – any waveform can be constructed by adding together a unique series of sine-waves, each specified by frequency, amplitude and phase ...
  – but an infinite number may be needed.

• Fourier analysis
  – Any waveform can be decomposed into a unique set of component sinusoids
  – involves complex mathematics but this is easily carried out by computers and digital signal processors.

• Periodic waves have spectra that can only consist of components at harmonic frequencies of the fundamental.

• Aperiodic waves can have anything else – almost always continuous spectra.
The BIG idea: Illustrated
Representing systems in terms of what they do to sinusoids:
Frequency responses
Characterisation of LTI-Systems

Input signal \rightarrow \text{SYSTEM} \rightarrow \text{Output signal}

Frequency

output amplitude
input amplitude

Amplitude

Frequency

Response

Frequency

Amplitude

Frequency
Characterisation of LTI-Systems

Input signal $\rightarrow$ SYSTEM $\rightarrow$ Output signal

transfer function
or frequency response

Amplitude $\rightarrow$ Frequency

Response $\rightarrow$ Frequency

Amplitude $\rightarrow$ Frequency
Amplitude Response: Key points

- Change made by system to amplitude of a sinewave – specified over a range of frequencies.
- Response = output amplitude/input amplitude
- Usually scaled in dB as:
  \[ 20 \times \log_{10}\left(\frac{\text{output amplitude}}{\text{input amplitude}}\right) = \text{response (dB re input amplitude)} \]
Filters

- Common name for systems that change amplitude and/or phase of waves
  - or just any LTI system
- Simple filters – low-pass and high-pass
An ideal low-pass filter

- Sudden change from gain of 1 to a very small value (virtually no output at all) at cut-off frequency $f_c$
A realistic low-pass filter

- Defined as frequency where gain is -3dB.
- -3 dB is equivalent to half-power not half-amplitude
  $10 \log(0.5) = -3.0$
Lowpass filters can vary in the steepness of their slopes.
Slope of filter

- Often constant in dB for a given frequency ratio
  - e.g., −6 dB per octave (doubling of frequency)
- suggests the use of a log frequency scale as well as a log amplitude ratio scale
  - dB in log base 10 (10, 100, 1000, etc.)
  - octave scale is log base 2, as implied in the frequency scale of an audiogram (125, 250, 500, 1000, 2000, etc).
Filter slope – in dB/octave

- Degrees of steepness of slope less than 18 dB/octave can be called "shallow"
- 48 dB/octave or more can be called "steep"
High-pass filters
Simple filters: Key points

- High-pass or low-pass characteristics
- Defined by:
  - cut-off frequency and slope of response
- Almost all natural sounds a mixture of frequencies
Systems in cascade

- Each stage acts independently, on the output of the previous stage.
Systems in cascade

• On a linear response scale:
  – Overall amplitude response is *product* of component responses (e.g., multiply the amplitude responses)

• On a dB (logarithmic) response scale
  – Overall amplitude response is the *sum* of the component responses (i.e., sum the amplitude responses) ...
  – Because taking logarithms turns multiplication into addition
Describing the width of a band-pass filter

-3 dB gain point

- Here bandwidth (BW) is 150 Hz
Natural filters

• Pendulum
• A relevant acoustic example:
  – a cylinder or tube closed at one end and open at the other
  – *e.g.* the ear canal
The ear canal
An acoustic tube closed at one end and open at the other (≈23 mm long)
Resonance

- Tubes like the ear canal form a special type of simple filter ...  
  - a resonator – similar to a band-pass filter
- Response not defined by independent high-pass and low-pass cutoff frequencies, but from a single centre frequency (the resonant frequency)
  - Resonant frequency is determined by physical characteristics of the system, often to do with size.
  - Bandwidth measured at 3 dB down points ...
  - determined by the damping in the system
  - more damping=broader bandwidth
What is damping?

• The loss of energy in a vibrating system, typically due to frictional forces
• A child on a swing: feet up or brushing the floor
• A pendulum with or without a cone over the bob.
• An acoustic resonator (like the ear canal) with or without gauze over its opening
• But all systems have some damping, even if just from molecules moving against one another